Exercise 76

For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?

Solution

Differentiate the given function using the chain rule.

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{d}{dx}(e^{rx}) \\ &= e^{rx} \cdot \frac{d}{dx}(rx) \\ &= e^{rx} \cdot (r) \\ &= re^{rx} \end{aligned}$$

Take another derivative.

$$\begin{aligned} y'' &= \frac{d}{dx}(y') \\ &= \frac{d}{dx}(re^{rx}) \\ &= re^{rx} \cdot \frac{d}{dx}(rx) \\ &= re^{rx} \cdot (r) \\ &= r^2e^{rx} \end{aligned}$$

Now plug these formulas into the differential equation.

$$\begin{aligned} y'' - 4y' + y &= (r^2e^{rx}) - 4(re^{rx}) + (e^{rx}) \\ &= (r^2 - 4r + 1)e^{rx} \end{aligned}$$

In order for the right side to be zero, the quantity in parentheses must be zero.

$$r^2 - 4r + 1 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

The values of r are therefore

$$r = \{2 - \sqrt{3}, 2 + \sqrt{3}\}.$$